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Abstract

Conditioned on the generating functions of offspring distribution, we study the asymptotic behaviour of the probability of non-extinction of a critical multi-type Galton-Watson process in i.i.d. random environments by using limits theorems for products of positive random matrices. Under some certain assumptions, the survival probability is proportional to $1/\sqrt{n}$.

Keywords: multi-type branching process, survival probability, random environment, product of matrices, critical case.
AMS classification 60J80, 60F17, 60K37.

1 Introduction and main results

Many researchers study the behaviour of critical branching processes in random environment. In 1999, under some strongly restricted conditions, Dyakonova [2] studied the multi-type case using the similar tools of one-type case. In 2002, Geiger achieved an important result for critical one-type case in random i.i.d. environment, see [4]. In the present work, we propose a variation of Dyakonova's result by imitating the approach of Geiger and Kersting [5].

Fix an integer $p \geq 2$ and denote $\mathbb{R}^p$ the set of p-dimensional column vectors with real coordinates ; for any column vector $x = (x_i)_{1 \leq i \leq p} \in \mathbb{R}^p$, we denote $\tilde{x}$ the row vector $\tilde{x} := (x_1, \ldots, x_p)$. Let $\mathbf{1}$ be the column vector of $\mathbb{R}^p$ where all coordinates equal 1. We fix a basis $\{e_i, 1 \leq i \leq p\}$ in $\mathbb{R}^p$ and denote $|.|$ the corresponding L_1 norm. Denote $\mathbb{N}^p$ the set of all p-dimensional column vectors whose components are non-negative integers. We also consider the general linear semi -group S^+ of $p \times p$ matrices with non-negative coefficients. We endow S^+ with the L_1-norm denoted also by $|.|$.

The multi-type Galton-Watson process is a temporally homogeneous vector Markov process $Z_0, Z_1, Z_n, \ldots$, whose states are column vectors in $\mathbb{N}^p$. We always assume that Z_0 is non-random. For any $1 \leq i \leq p$, the i-th component $Z_n(i)$ of Z_n may be interpreted as the number of objects of type i in the n-th generation.

We consider a family $\{f_\xi : \xi \in \mathbb{R}\}$ of multi-variate probability generating functions $f_\xi(s) = (f_\xi^{(i)}(s))_{1 \leq i \leq p}$ where

$$f_\xi^{(i)}(s) = \sum_{\alpha \in \mathbb{N}^p} p_\xi^{(i)}(\alpha) s^\alpha$$

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with

1. $\alpha = (\alpha_i)_i \in \mathbb{N}^p, s = (s_i)_i, 0 \leq s_i \leq 1$ for $i = 1, \ldots, p$ and $s^\alpha = s_1^{\alpha_1} \ldots s_p^{\alpha_p}$;
2. $p_\xi^{(i)}(\alpha) = p_\xi^{(i)}(\alpha_1, \ldots, \alpha_p)$ is the probability that an object of type i in environment ξ has α_1 children of type $1, \ldots, \alpha_p$ children of type p.

Let $\xi = \{\xi_n, n = 0, 1, \ldots\}$ be a sequence of real valued i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Galton-Watson process with p types of particles in a random environment ξ describes the evolution of a particle population $Z_n = (Z_n(1), \ldots, Z_n(p))$ for $n = 0, 1, \ldots$.
We assume that for $\xi \in \mathbb{R}$ and $i = 1, \ldots, p$, if $\xi_n = \xi$, then each of the $Z_i(n)$ particles of type i, existing at time n produces offspring in accordance with the p-dimensional generating function $f_\xi^{(i)}(s)$ independently of the reproduction of other particles of all types.

If $Z_0 = e_i$ then Z_1 has the generating function:

$$f_{\xi_0}^{(i)}(s) = \sum_{\alpha \in \mathbb{N}^p}^{+\infty} p_{\xi_0}^{(i)}(\alpha) s^\alpha.$$

In general, if $Z_n = (\alpha_1, \ldots, \alpha_p)$, then Z_{n+1} is the sum of $\alpha_1 + \ldots + \alpha_p$ independent random vectors where α_i particles of type i have the generating function $f_{\xi_n}^{(i)}$ for $i = 1, \ldots, p$. It is obvious that if $Z_n = 0$, then $Z_{n+1} = 0$.

Denote $f_n = f_{\xi_n}$. By the above descriptions, (written in equation 2.1 in [10]) for any $s = (s_i)_i, 0 \leq s_i \leq 1$

$$\mathbb{E}\left[s^{Z_n} | Z_0, ..., Z_{n-1}, f_0, ..., f_{n-1}\right] = f_{n-1}(s)^{Z_{n-1}}$$

which yields (lemma 2.1 in [10])

$$\mathbb{E}\left[s^{Z_n} | f_0^{(i)}, ..., f_{n-1}\right] := \mathbb{E}\left[s^{Z_n} | Z_0 = e_i, f_0, ..., f_{n-1}\right] = f_0^{(i)}(f_1(...f_{n-1}(s)...)).$$

In particular, the probability of non-extinction $q_n^{(i)}$ at generation n given the environment $f_0^{(i)}, f_1, ...f_{n-1}$ is

$$\begin{aligned} q_n^{(i)} &:= \mathbb{P}(Z_n \neq 0 | f_0^{(i)}, ..., f_{n-1}) \\ &= 1 - f_0^{(i)}(f_1(...f_{n-1}(0)...)) = \tilde{e}_i(\mathbf{1} - f_0(f_1(...f_{n-1}(0)...))), \end{aligned} \qquad (1)$$

so that

$$\mathbb{E}[q_n^{(i)}] = \mathbb{E}[\mathbb{P}(Z_n \neq 0 | f_0^{(i)}, ..., f_{n-1})] = \mathbb{P}(Z_n \neq 0 | Z_0 = e_i).$$

As in the classical one-type case, the asymptotic behaviour of the quantity above is controled by the mean of the offspring distributions. From now on, we assume that the offspring distributions have finite first and second moments; the generating functions $f_\xi^{(i)}, \xi \in \mathbb{R}, 1 \leq i \leq p$, are thus C^2-functions on $[0, 1]^p$ and we introduce

1. the random mean matrices $M_{\xi_n} = (M_{\xi_n}(i,j))_{1\leq i,j\leq p} = \left(\dfrac{\partial f^{(i)}_{\xi_n}(1)}{\partial s_j}\right)_{i,j}$ of the vector-valued random generating function $f_{\xi_n}(s)$ at $s = \mathbf{1}$, namely

$$M_{\xi_n} = \begin{pmatrix} \dfrac{\partial f^{(1)}_{\xi_n}(\mathbf{1})}{\partial s_1} & \cdots & \dfrac{\partial f^{(1)}_{\xi_n}(\mathbf{1})}{\partial s_p} \\ \vdots & & \\ \dfrac{\partial f^{(p)}_{\xi_n}(\mathbf{1})}{\partial s_1} & \cdots & \dfrac{\partial f^{(p)}_{\xi_n}(\mathbf{1})}{\partial s_p} \end{pmatrix},$$

where $\mathbf{1} = (1, ...1)^T$.

2. the random Hessian matrices $B^{(i)}_{\xi_n} = (B^{(i)}_{\xi_n}(k,l))_{1\leq k,l\leq p} = \left(\frac{\partial^2 f^{(i)}_{\xi_n}}{\partial s_k \partial s_l}(\mathbf{1})\right)_{k,l}, 1 \leq i \leq p$, of the real-valued random generating function $f^{(i)}_{\xi_n}(s)$ at $s = \mathbf{1}$.

The random variables M_{ξ_n} and $B^{(i)}_{\xi_n}$ are i.i.d.. The common law of the M_{ξ_n} is denoted by μ and for the sake of brevity, we write M_n instead of M_{ξ_n}.

Let R_n and L_n denote the right and the left product of random matrices $M_k, k \geq 0$, respectively $R_n = M_0 M_1 ... M_{n-1}$ and $L_n = M_{n-1} ... M_1 M_0$.

By [3], if $\mathbb{E}(\max(0, \ln|M_0|)) < +\infty$, then the sequence $\left(\dfrac{1}{n}\ln|R_n|\right)_n$ converges $\mathbb{P}$-almost surely to some constant limit $\pi := \lim\limits_{n\to+\infty} \dfrac{1}{n}\mathbb{E}[\ln|R_n|]$. Furthermore, by [10], if there exists a constant $A > 0$ such that $\dfrac{1}{A} \leq M_{\xi_n}(i,j) \leq A$ and $0 \leq B^{(i)}_{\xi_n}(k,l) \leq A$ $\mathbb{P}$-almost surely for any $1 \leq i,j,k,l \leq p$, then the process $(Z_n)_n$ extincts $\mathbb{P}$-almost surely if and only if $\pi \leq 0$.

In the present work, we will focus our attention on the so-called **critical case**, that is $\pi = 0$, and precise the speed of extinction of the Galton-Watson process.

We define the cone $\mathcal{C}$, the sphere $\mathbb{S}^{p-1}$ and the space $\mathbb{X}$ respectively as follows:

$$\mathcal{C} = \{\tilde{x} = (x_1, ..., x_p) \in \mathbb{R}^p : \forall i = 1, ..., p,\ x_i \geq 0\},$$

$\mathbb{S}^{p-1} = \{\tilde{x} : x \in \mathbb{R}^p, |x| = 1\}$, and $\mathbb{X} = \mathcal{C} \cap \mathbb{S}^{p-1}$.

The semi-group S^+ acts on $\mathbb{X}$ by the projective action defined by: $\tilde{x} \cdot g = \dfrac{\tilde{x}g}{|\tilde{x}g|}$ for $\tilde{x} \in \mathbb{X}$ and $g \in S^+$. On the product space $\mathbb{X} \times S^+$ we define the function ρ by setting $\rho(\tilde{x}, g) := \log|\tilde{x}g|$ for $(g, \tilde{x}) \in \mathbb{X} \times S^+$. This function satisfies the cocycle property, namely for any $g, h \in S^+$ and $\tilde{x} \in \mathbb{X}$,

$$\rho(\tilde{x}, gh) = \rho(\tilde{x} \cdot g, h) + \rho(\tilde{x}, g). \tag{2}$$

Under conditions H1–H3 which are introduced below, there exists a unique μ-invariant measure ν on $\mathbb{X}$ such that, for any continuous function φ on $\mathbb{X}$,

$$(\mu * \nu)(\varphi) = \int_{S^+}\int_{\mathbb{X}} \varphi(\tilde{x} \cdot g)\nu(d\tilde{x})\mu(dg) = \int_{\mathbb{X}} \varphi(\tilde{x})\nu(d\tilde{x}) = \nu(\varphi).$$

Moreover, the upper Lyapunov exponent π defined above coincides with the quantity $\int_{\mathbb{X}\times S^+} \rho(\tilde{x}, g)\mu(dg)\nu(d\tilde{x})$ and is finite [1].

In the sequel, we first focus our attention to the class $\mathcal{H}$ of linear-fractional multi-dimensional generating functions f_ξ which contains functions of the form

$$f_\xi(s) = \mathbf{1} - \frac{1}{1+\tilde{\gamma}_\xi(\mathbf{1}-s)} M_\xi(\mathbf{1}-s),$$

where $\tilde{\gamma}_\xi = (\gamma_\xi, ..., \gamma_\xi) \in \mathbb{R}^p$ with $\gamma_\xi > 0$.

Hypotheses H: the variables f_ξ are $\mathcal{H}$-valued and γ_ξ (resp. the distribution μ of the M_ξ) satisfies hypothesis H0 (resp. H1–H5), with

H0. There exists a real positive number A such that $\frac{1}{A} \leq \gamma_\xi \leq A$ $\mathbb{P}$-almost surely.

H1. There exists $\epsilon_0 > 0$ such that $\int_{S^+} |g|^{\epsilon_0} \mu(dg) < \infty$.

H2. (Strong irreducibility). The support of μ acts strongly irreducibly on $\mathbb{R}^p$, i.e. no proper finite union of subspaces of $\mathbb{R}^p$ is invariant with respect to all elements of the semi-group it generates.

H3. There exists a real positive number B such that, μ-almost surely, for any and $i, j, k, l \in \{1, ..., p\}$:

$$\frac{1}{B} \leq \frac{M(i,j)}{M(k,l)} \leq B.$$

H4. The upper Lyapunov exponent of the distribution μ is equal to 0.

H5. There exists $\delta > 0$ such that $\mu\{g \in G \mid \forall \tilde{x} \in \mathcal{C},\ |x| = 1,\ \ln|\tilde{x}g| \geq \delta\} > 0$.

We now state the main result of this paper.

Theorem 1.1 *Under hypotheses* **H**, *for any* $i \in \{1, ..., p\}$, *there exists a real number* $\beta_i \in (0, +\infty)$ *such that*

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}(Z_n \neq 0 | Z_0 = e_i) = \beta_i.$$

When the f_ξ are not assumed to be linear fractional generating functions, we have the following weaker result:

Theorem 1.2 *Assume that the* f_ξ *are* C^2*-functions on* $[0,1]^p$ *such that*

1. *there exists* $A > 0$ *such that, for any* $i, k, l \in \{1, \ldots, p\}$

$$\frac{\partial^2 f_\xi^{(i)}}{\partial s_k \partial s_l}(\mathbf{1}) \leq A \frac{\partial f_\xi^{(i)}}{\partial s_k}(\mathbf{1}),$$

2. *the distribution* μ *of the* $M_\xi = \left(\frac{\partial f_\xi^{(i)}}{\partial s_j}(\mathbf{1})\right)_{1\leq i,j\leq p}$ *satisfies hypotheses H1–H5.*

Then, there exist real constants $0 < c_1 < c_2 < +\infty$ *such that, for any* $i \in \{1, \ldots, p\}$, *and* $n \geq 1$

$$\frac{c_1}{\sqrt{n}} \leq \mathbb{P}(Z_n \neq 0 | Z_0 = e_i) \leq \frac{c_2}{\sqrt{n}}. \qquad (3)$$

In particular, under weaker assumptions than Kaplan [10], this theorem states that the process $(Z_n)_{n\geq 0}$ extincts $\mathbb{P}$-a.s. in the critical case.

Notations. *Let* $c > 0$; *we shall write* $f \overset{c}{\preceq} g$ *(or simply* $f \preceq g$*) when* $f(x) \leq cg(x)$ *for any value of* x. *The notation* $f \overset{c}{\asymp} g$ *(or simply* $f \asymp g$*) means* $f \overset{c}{\preceq} g \overset{c}{\preceq} f$.

2 Preliminary concepts

From now on, we fix $B \geq 1$ and denote $S = S(B)$ the semi-group generated by matrices $g = (g_{i,j})_{i,j}$ in S^+ satisfying the condition

$$g_{i,j} \overset{B}{\asymp} g_{k,l} \quad \text{for all} \, 1 \leq i, j, k, l \leq p.$$

2.1 Product of matrices with non-negative coefficients

We describe in this section some properties of the set S^+. We first endow $\mathbb{X}$ with a distance d which is a variant of the Hilbert metric; it is bounded on $\mathbb{X}$ and any element $g \in S^+$ acts on $(\mathbb{X}, d)$ as a contraction; we summarise here its construction and its major properties.

For any $x, y \in \mathbb{X}$, we write

$$m(x, y) = \min\left\{ \frac{x_i}{y_i} \middle| i = 1, \ldots, p \text{ such that } y_i > 0 \right\}$$

and we set

$$d(x, y) := \varphi\Big(m(x, y)m(y, x)\Big)$$

where φ is the one-to-one function on $[0, 1]$ defiend by $\varphi(s) := \dfrac{1-s}{1+s}$. For $g \in S^+$, set

$$c(g) := \sup\{d(g \cdot x, g \cdot y) \mid x, y \in \mathbb{X}\}.$$

We now present some crucial properties of d.

Proposition 2.1 *The function d is a distance on $\mathbb{X}$ wich satisfies the following properties:*

1. $\sup\{d(x, y) \mid x, y \in \mathbb{X}\} = 1.$
2. *for any* $g = (g_{ij})_{i,j} \in S^+$

   $$c(g) = \max_{i,j,k,l \in \{1,\ldots,p\}} \frac{|g_{ij}g_{kl} - g_{il}g_{kj}|}{g_{ij}g_{kl} + g_{il}g_{kj}}.$$

   In particular, there exists $\kappa \in [0, 1)$ which depends on B such that $c(g) \leq \kappa < 1$ for any $g \in S(B)$.
3. $d(g \cdot x, g \cdot y) \leq c(g)d(x, y) \leq c(g)$ *for any* $x, y \in \mathbb{X}$ *and* $g \in S(B)$.
4. $c(gg') \leq c(g)c(g')$ *for any* $g, g' \in S(B)$.

The following lemma is crucial in the sequel to control the asymptotic behaviour of the norm of products of matrices of $S(B)$.

Lemma 2.2 *Under hypothesis H3, for any $g, h \in S(B)$, and $1 \leq i, j, k, l \leq p$*

$$g(i, j) \overset{B^2}{\asymp} g(k, l), \text{obtained from [3]}. \tag{4}$$

In particular, there exist $c > 1$ such that for any $g \in S(B)$ and for any $\tilde{x}, \tilde{y} \in \mathbb{X}$,

1. $|gx| \overset{c}{\asymp} |g|$ *and* $|\tilde{y}g| \overset{c}{\asymp} |g|$,
2. $|\widetilde{y}gx| \overset{c}{\asymp} |g|$,
3. $|gh| \overset{c}{\asymp} |g||h|$.

2.2 Conditioned product of random matrices

Recall that $(M_n)_{n\geq 0}$ is a sequence of i.i.d. matrices whose law μ satisfies hypothese H and $R_n = M_0 ... M_{n-1}$ for $n \geq 1$. Consider the homogenous Markov chain $(X_n)_n$ on $\mathbb{X}$, with initial value $X_0 = \tilde{x} \in \mathbb{X}$, defined by

$$X_n = \tilde{x} \cdot R_n, n \geq 1.$$

Its transition probability P is given by: for any $\tilde{x} \in \mathbb{X}$ and any bounded Borel function $\varphi : \mathbb{X} \to \mathbb{R}$,

$$P\varphi(\tilde{x}) := \int_{S^+} \varphi(\tilde{x} \cdot g)\mu(dg).$$

The chain $(X_n)_{n\geq 0}$ has been the object of many studies, in particular there exists on $\mathbb{X}$ a unique P-invariant probability measure ν. Indeed, by Proposition 2.1, for any $\tilde{x}, \tilde{y} \in \mathbb{X}$, one gets

$$d(\tilde{x} \cdot L_n, \tilde{y} \cdot L_n) \leq \kappa^n \tag{5}$$

so that $\sup_{k\geq 0} d(\tilde{x} \cdot L_{n+k}, \tilde{x} \cdot L_n) \to 0$ a.s. as $n \to +\infty$; the sequence $(\tilde{x} \cdot L_n)_{n\geq 0}$ thus converges a.s. to some $\mathbb{X}$-valued random variable Z. It follows that the Markov chain $(\tilde{x} \cdot R_n)_{n\geq 0}$ converges in distribution to the law ν of Z, which is the unique P-invariant probability measure on $\mathbb{X}$. Property 5 allows to prove that the restriction of P to some suitable space of continuous functions from $\mathbb{X}$ to $\mathbb{C}$ is quasi-compact, which is a crucial ingredient to study the asymptotic behavior of $(\tilde{x} \cdot R_n)_{n\geq 0}$ ([8], [1], [9]).

In the sequel, we deal with the random process $(S_n)_n$ defined by $S_0 = S_0(\tilde{x}, a) := a, S_n = S_n(\tilde{x}, a) := a + \ln |\tilde{x} R_n|$, where $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$. Iterating the cocycle property (2),the basic representation of $S_n(\tilde{x}, a)$ arrives:

$$S_n(\tilde{x}, a) = a + \ln |\tilde{x} R_n| = a + \sum_{k=0}^{n-1} \rho(X_k, M_k). \tag{6}$$

Let $m_n = m_n(\tilde{x}) := \min(S_0(\tilde{x}), ..., S_n(\tilde{x}))$ be the successive minima of the sequence $(S_n(\tilde{x}))_n$ and for $a \geq 0$ denote $\mathbf{m}_n(\tilde{x}, a) := \mathbb{P}_{\tilde{x},a}[m_n > 0]$. Let us emphasize that for any $a \in \mathbb{R}$ the sequence $(X_n, S_n)_n$ is a Markov chain on $\mathbb{X} \times \mathbb{R}$ whose transition probability $\widetilde{P}$ is defined by: for any $(\tilde{x}, a) \in \mathbb{X} \times \mathbb{R}$ and any bounded Borel function $\psi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$

$$\widetilde{P}\psi(\tilde{x}, a) = \int_{S^+} \psi(\tilde{x} \cdot g, a + \rho(\tilde{x}, g))\mu(dg).$$

We denote $\widetilde{P}_+$ the restriction of $\widetilde{P}$ to $\mathbb{X} \times \mathbb{R}^+_*$ defined by: for $a > 0$ and any $\tilde{x} \in \mathbb{X}$

$$\widetilde{P}_+((\tilde{x}, a), \cdot) = 1_{\mathbb{X}\times\mathbb{R}^+_*}(\cdot)\widetilde{P}((\tilde{x}, a), \cdot).$$

From now on, fix $a > 0$ and denote by τ the first time the random process $(S_n)_n$ becomes non-positive:

$$\tau := \min\{n \geq 1 : S_n \leq 0\}.$$

For any $\tilde{x} \in \mathbb{X}$ and $a > 0$, let us denote $\mathbb{P}_{\tilde{x},a}$ the probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned to the event $[X_0 = \tilde{x}, S_0 = a]$ and $\mathbb{E}_{\tilde{x},a}$ the corresponding expectation; we omit the index a when $a = 0$ and denote $\mathbb{P}_{\tilde{x}}$ the corresponding probability.

We now present a general result concerning the behavior of the tail distribution of the random variable τ; we refer to [6] in the case of product of random invertible matrices and under general suitable conditions we do not present here. The statement below is given in the case of products of matrices with non-negative coefficients, it is not a direct consequence of [6] but the proof is the same, we postpone the sketch of its main steps in the Appendix.

Under hypotheses H1–H5, the function $h : \mathbb{X} \times \mathbb{R}_*^+ \to \mathbb{R}_*^+$ defined by

$$h(\tilde{x}, a) = \lim_{n \to +\infty} \mathbb{E}_{\tilde{x},a} [S_n; \tau > n] \tag{7}$$

is $\widetilde{P}_+$–Harmonic, namely $\mathbb{E}_{\tilde{x},a} [h(X_1, S_1); \tau > 1] = h(\tilde{x}, a)$ for any $\tilde{x} \in \mathbb{X}$ and $a > 0$. Furthermore, there exists $c > 0$ such that

$$\forall \tilde{x} \in \mathbb{X}, \forall a > 0 \qquad h(\tilde{x}, a) \leq c(1 + a) \tag{8}$$

and the function $a \mapsto h(\tilde{x}, a)$ is increasing on $\mathbb{R}_*^+$.

The tail of the distribution of τ is given by the following theorem; the relation $u_n \sim v_n$ defines $\lim_{n \to +\infty} \frac{u_n}{v_n} = 1$.

Theorem 2.3 *Assume hypotheses H1–H5. For any $\tilde{x} \in \mathbb{X}$ and $a > 0$,*

$$\mathbb{P}_{\tilde{x},a}(\tau > n) \sim \frac{2}{\sigma\sqrt{2\pi n}} h(\tilde{x}, a) \text{ as } n \to \infty, \tag{9}$$

where $\sigma^2 > 0$ is the variance of the Markov walk $(S_n)_n$, given in [6]. Moreover, there exists a constant $c > 0$ such that for any $\tilde{x} \in \mathbb{X}$, $a > 0$ and $n \geq 0$

$$\sqrt{n}\mathbb{P}_{\tilde{x},a}(\tau > n) \leq c(1 + a). \tag{10}$$

Remark. The fact that $\sigma^2 > 0$ is a direct consequence of hypotheses H2 and H5 (which implies in particular that the semi-group generated by the support of μ is unbounded); see [1], chap 6, Lemmas 5.2 and 5.3 and section 8 for the details.

3 Proof of Theorem 1.1

3.1 Expression of non-extinction probability

For any $0 \leq k < n$ and $\tilde{x} \in \mathbb{X}$, set $R_{k,n} := M_k ... M_{n-1}$ and $R_{k,n} := I$ otherwise. Let $Y_{k,n}(x) := R_{k,n} \cdot x$; the sequence $(Y_{k,n}(x))_n$ converges $\mathbb{P}$-almost surely to some limit $Y_{k,\infty}$ which does not depend on x, see [3]. Hypothesis H and (1) yield

$$(q_n^{(i)})^{-1} = \frac{1 + \tilde{\gamma}_0 M_1 \ldots M_{n-1}\mathbf{1} + \tilde{\gamma}_1 M_2 \ldots M_{n-1}\mathbf{1} + \ldots + \tilde{\gamma}_{n-1}\mathbf{1}}{\tilde{e}_i R_n \mathbf{1}}.$$

Indeed, recall that $f_\xi(s)$ are linear-fractional generating functions, it is obvious that

$$\begin{aligned}
\mathbf{1} - f_0(f_1(...f_{n-1}(s)...)) &= \frac{M_0(\mathbf{1} - f_1(...f_{n-1}(s)...))}{1 + \tilde{\gamma}_0(\mathbf{1} - f_1(...f_{n-1}(s)...))} \\
&= \frac{M_0 M_1(\mathbf{1} - f_2(...f_{n-1}(s)...))}{1 + \tilde{\gamma}_0 M_0(\mathbf{1} - f_2(...f_{n-1}(s)...)) + \tilde{\gamma}_1(\mathbf{1} - f_2(...f_{n-1}(s)...))} \\
&= \ldots \\
&= \frac{M_0 ... M_{n-1}(\mathbf{1} - s)}{1 + \tilde{\gamma}_0 M_1 ... M_{n-1}(\mathbf{1} - s) + \tilde{\gamma}_1 M_2 ... M_{n-1}(\mathbf{1} - s) + ... + \tilde{\gamma}_{n-1}(\mathbf{1} - s)}
\end{aligned}$$

Substituting $s = 0$, the expression of $q_n^{(i)}$ arrives.

In other words, since we have $\tilde{e}_i R_k R_{k,n}\mathbf{1} = \tilde{e}_i M_0 \dots M_{n-1}\mathbf{1}$ for any $1 \le k \le n$, we may write

$$\begin{aligned}(q_n^{(i)})^{-1} &= \frac{1}{\tilde{e}_i R_n \mathbf{1}} + \sum_{k=0}^{n-1} \frac{\tilde{\gamma}_k Y_{k+1,n}}{\tilde{e}_i R_k Y_{k+1,n}} \\ &= \frac{1}{\tilde{e}_i R_n \mathbf{1}} + \sum_{k=0}^{n-1} \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,n}}. \end{aligned} \tag{11}$$

In the sequel, we prove that the sequence $(q_n^{(i)})_{n\ge 1}$ converges almost surely to a finite quantity $q_\infty^{(i)}$ given by

$$(q_\infty^{(i)})^{-1} = \sum_{k=0}^{+\infty} \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}}, \tag{12}$$

with respect to a new probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ introduced in the following subsection (Lemma 3.2).

3.2 Construction of a new probability measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ conditioned to the environment

Since the function h is $\widetilde{P}_+$–Harmonic on $\mathbb{X} \times \mathbb{R}_*^+$, it gives rise to a Markov kernel $\widetilde{P}_+^h$ on $\mathbb{X} \times \mathbb{R}_*^+$ defined by

$$\widetilde{P}_+^h \phi = \frac{1}{h}\widetilde{P}_+(h\phi)$$

for any bounded measurable function ϕ on $\mathbb{X} \times \mathbb{R}_*^+$. The kernels $\widetilde{P}_+$ and $\widetilde{P}_+^h$ are related to the stopping times τ by the following identity: for any $\tilde{x} \in \mathbb{X}$, $a > 0$ and $n \ge 1$,

$$\begin{aligned}(\widetilde{P}_+^h)^n \phi(\tilde{x}, a) &= \frac{1}{h(\tilde{x}, a)} \widetilde{P}_+^n (h\phi)(\tilde{x}, a) \\ &= \frac{1}{h(\tilde{x}, a)} \mathbb{E}_{\tilde{x},a}\left[h\phi(X_n, S_n); \tau > n\right] \\ &= \frac{1}{h(\tilde{x}, a)} \mathbb{E}_{\tilde{x},a}\left[h\phi(X_n, S_n); m_n > 0\right].\end{aligned}$$

This new Markov chain with kernel $\widetilde{P}_+^h$ allows us to change the measure on the canonical path space $((\mathbb{X} \times \mathbb{R})^{\otimes \mathbb{N}}, \sigma(X_n, S_n : n \ge 0), \theta)$ of the Markov chain $(X_n, S_n)_{n\ge 0}$ [4] from $\mathbb{P}$ to the measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ characterized by the property that

$$\widehat{\mathbb{E}}_{\tilde{x},a}[\varphi(X_0, S_0, ..., X_k, S_k)] = \frac{1}{h(\tilde{x}, a)} \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, ..., X_k, S_k) h(X_k, S_k);\ m_k > 0] \tag{13}$$

for any positive Borel function φ on $(\mathbb{X} \times \mathbb{R})^{k+1}$ that depends on $X_0, S_0, ..., X_k, S_k$.

[4] θ denotes here the shift operator on $(\mathbb{X}\times\mathbb{R})^{\otimes\mathbb{N}}$ defined by $\theta\Big((x_k, s_k)_k\Big) = (x_{k+1}, s_{k+1})_k$ for any $(x_k, s_k)_k$ in $(\mathbb{X} \times \mathbb{R})^{\otimes\mathbb{N}}$

For any $0 \le k \le n$,

$$
\begin{aligned}
& \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k) | m_n > 0] \\
= \quad & \frac{1}{\mathbb{P}_{\tilde{x},a}(m_n > 0)} \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k); S_0 > 0, S_1 > 0, \ldots, S_n > 0] \\
= \quad & \frac{1}{\mathbb{P}_{\tilde{x},a}(m_n > 0)} \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k); S_0 > 0, a + \rho(X_0, M_0) > 0, \\
& \qquad \ldots, a + \sum_{i=0}^{k-1} \rho(X_i, M_i) + \sum_{i=k}^{n-1} \rho(X_i, M_i) > 0] \\
= \quad & \frac{1}{\mathbb{P}_{\tilde{x},a}(m_n > 0)} \mathbb{E}_{\tilde{x},a}[\mathbb{E}[\varphi(X_0, S_0, \ldots, X_k, S_k); S_0 > 0, \ldots, S_k > 0, \\
& \qquad S_0 \circ \theta^k > 0, \ldots, S_k + S_{n-k} \circ \theta^k > 0 | \sigma(M_0, ..., M_{k-1})]] \\
= \quad & \frac{1}{\mathbb{P}_{\tilde{x},a}(m_n > 0)} \mathbb{E}_{\tilde{x},a}\Big[\varphi(X_0, S_0, \ldots, X_k, S_k) \\
& \qquad \mathbb{E}[S_k > 0, \ldots, S_k + S_{n-k} \circ \theta^k > 0 | \sigma(M_0, .., M_{k-1})]; m_k > 0\Big] \\
= \quad & \frac{1}{\mathbb{P}_{\tilde{x},a}(m_n > 0)} \mathbb{E}_{\tilde{x},a}\Big[\varphi(X_0, S_0, \ldots, X_k, S_k) \\
& \qquad \mathbb{P}_{X_k, S_k}(S_0 \circ \theta^k > 0, \ldots, S_{n-k} \circ \theta^k > 0); m_k > 0\Big] \\
= \quad & \frac{1}{\mathbb{P}_{\tilde{x},a}(\tau > n)} \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k) \mathbb{P}_{X_k, S_k}(\tau > n-k); m_k > 0].
\end{aligned}
$$

Hence,

$$
\mathbb{E}_{\tilde{x},a}\left[\varphi(X_0, S_0, \ldots, X_k, S_k); m_n > 0\right] = \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k) \mathbf{m}_{n-k}(X_k, S_k); m_k > 0]. \tag{14}
$$

Moreover, in view of Theorem 2.3, the dominated convergence theorem and (14), we obtain for any bounded function φ with compact support,

$$
\begin{aligned}
& \lim_{n \to +\infty} \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k) | m_n > 0] \\
= \quad & \frac{1}{h(\tilde{x}, a)} \mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k) h(X_k, S_k); m_k > 0] \\
= \quad & \widehat{\mathbb{E}}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k)],
\end{aligned} \tag{15}
$$

which clarifies the interpretation of $\widehat{\mathbb{P}}_{\tilde{x},a}$. Using [6], it follows that

$$
\mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, \ldots, X_k, S_k) | m_n > 0] \sim \frac{\sqrt{n}}{\sqrt{n-k}} \frac{\mathbb{E}_{\tilde{x},a}[\varphi(X_0, S_0, ..., X_k, S_k) h(X_k, S_k); m_k > 0]}{h(\tilde{x}, a)}.
$$

Hence when n tends to $+\infty$, (15) arrives.
Now we formalize in three steps the construction of a new probability measure, denoted again $\widehat{\mathbb{P}}_{\tilde{x},a}$, for each $\tilde{x} \in \mathbb{X}$ and $a > 0$, but defined this time on the bigger σ-algebra $\sigma(f_n, Z_n : n \ge 0)$. Retaining the notations from the previous parts, the measure $\widehat{\mathbb{P}}_{\tilde{x},a}$ is characterized by properties (13), (16) and (17).
Step 1. The marginal distribution of $\widehat{\mathbb{P}}_{\tilde{x},a}$ on $\sigma(X_n, S_n : n \ge 0)$ is $\widehat{\mathbb{P}}_{\tilde{x},a}$ characterized by the property (13).

Step 2. The conditional distribution of $(f_n)_{n\geq 0}$ under $\widehat{\mathbb{P}}_{\tilde{x},a}$ given $X_0 = \tilde{x}_0 = \tilde{x}, X_i = \tilde{x}_i, S_0 = s_0 = a, S_i = s_i, ...$ is

$$\begin{aligned}
&\widehat{\mathbb{P}}_{\tilde{x},a}(f_k \in A_k, 0 \leq k \leq n | X_i = \tilde{x}_i, S_i = s_i, i \geq 0) \\
&\qquad\qquad = \mathbb{P}(f_k \in A_k, 0 \leq k \leq n | X_i = \tilde{x}_i, S_i(\tilde{x}) = s_i, i \geq 0),
\end{aligned} \tag{16}$$

defined for almost all $(\tilde{x}_i)_i$ and $(s_i)_i$ with respect to the law of $((X_n)_n, (S_n)_n)$ under $\mathbb{P}$ (and also under $\widehat{\mathbb{P}}_{\tilde{x},a}$ since $\widehat{\mathbb{P}}_{\tilde{x},a}$ is absolutely continuous with respect to $\mathbb{P}$ on $\sigma((X_n)_{n\geq 0}, (S_n)_{n\geq 0})$), for any measurable set A_k.

Step 3. The conditional distribution of $(Z_n)_{n\geq 0}$ under $\widehat{\mathbb{P}}_{\tilde{x},a}$ given $f_0^{(i)}, f_1, ...$ is the same as under $\mathbb{P}$, namely

$$\begin{aligned}
\widehat{\mathbb{E}}_{\tilde{x},a}\left[s^{Z_n} | Z_0, ..., Z_{n-1}, f_0^{(i)}, ..., f_{n-1}\right] &= f_{n-1}(s)^{Z_{n-1}} \\
&= \mathbb{E}\left[s^{Z_n} | Z_0, ..., Z_{n-1}, f_0^{(i)}, ..., f_{n-1}\right].
\end{aligned} \tag{17}$$

3.3 Proof of Theorem 1.1

For any $\tilde{x} \in \mathbb{X}, a > 0$ and $i \in \{1, \ldots, p\}$ let us denote $\mathbb{P}^{(i)}_{\tilde{x},a}$ the probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$ conditioned to the event $[X_0 = \tilde{x}, S_0 = a, Z_0 = i]$ and $\mathbb{E}^{(i)}_{\tilde{x},a}$ the corresponding expectation.

We separate the proof in 4 steps.

1. Fix $\rho > 1$, $\tilde{x} \in \mathbb{X}$ and $a > 0$, we prove that the sequence $(\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0 | m_{\rho n} > 0))_{n\geq 0}$ converges as $n \to +\infty$ to $\lim\limits_{m\to+\infty} \widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_m \neq 0)$.

2. We identify the limit of the sequence $(\widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_m \neq 0))_{m\geq 0}$ and prove that it belongs to $\mathbb{R}^+_*$.

3. We get rid of ρ and prove the sequence $(\sqrt{n}\, \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n > 0))_{n\geq 0}$ converges in $\mathbb{R}^+_*$ as $n \to +\infty$, for any $a > 0$.

4. We achieve the assertion by letting $a \to +\infty$.

Step 1. Fix $0 \leq m \leq n$. Using (14), then conditioned on $\sigma(f_0^{(i)}, ..., f_{m-1})$, finally $1_{[m_m>0]}$ and $\mathbf{m}_{\rho n-m}(X_m, S_m)$ are measurable with respect to $\sigma(f_0^{(i)}, ..., f_{m-1})$, we may write

$$\begin{aligned}
\mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0, m_{\rho n} > 0) &= \mathbb{P}^{(i)}_{\tilde{x},a}\left[Z_m \neq 0, m_m > 0, \mathbf{m}_{\rho n-m}(X_m, S_m)\right] \\
&= \mathbb{E}_{\tilde{x},a}\left[\mathbb{E}(1_{[Z_m\neq 0]} 1_{[m_m>0]} \mathbf{m}_{\rho n-m}(X_m, S_m) \mid f_0^{(i)}, ..., f_{m-1})\right] \\
&= \mathbb{E}_{\tilde{x},a}\left[\mathbb{P}(Z_m \neq 0 | f_0^{(i)}, ..., f_{m-1}) 1_{[m_m>0]} \mathbf{m}_{\rho n-m}(X_m, S_m)\right] \\
&= \mathbb{E}_{\tilde{x},a}\left[q_m^{(i)}, m_{\rho n} > 0\right]
\end{aligned} \tag{18}$$

so that, by (15), since $0 \leq m \leq n$ is fixed

$$\lim_{n\to+\infty} \mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0 | m_{\rho n} > 0) = \lim_{n\to+\infty} \mathbb{E}_{\tilde{x},a}\left[q_m^{(i)} | m_{\rho n} > 0\right] = \widehat{\mathbb{E}}_{\tilde{x},a}\left[q_m^{(i)}\right] = \widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_m \neq 0). \tag{19}$$

To get the similar result with n instead of m, we write, for $0 \leq m \leq n$

$$\begin{aligned}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0 | m_{\rho n} > 0) &= \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, Z_m \neq 0 | m_{\rho n} > 0) \\ &= \mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0 | m_{\rho n} > 0) \\ &\qquad\qquad - \mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0, Z_n = 0 | m_{\rho n} > 0). \end{aligned} \tag{20}$$

The first term of the right side of (20) is controled by (19); for the the second term, we use the following Lemma.

Lemma 3.1 *For any $\rho > 1$, $\tilde{x} \in \mathbb{X}$ and $a > 0$,*

$$\limsup_{m \to +\infty} \limsup_{n \to +\infty} \mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0, Z_n = 0 | m_{\rho n} > 0) = 0.$$

Therefore, by taking limits over n and then m in (20), it follows that

$$\begin{aligned} & \limsup_{n \to +\infty} \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0 | m_{\rho n} > 0) \\ = \; & \limsup_{m \to +\infty} \limsup_{n \to +\infty} \mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0 | m_{\rho n} > 0) - \limsup_{m \to +\infty} \limsup_{n \to +\infty} \mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0, Z_n = 0 | m_{\rho n} > 0) \\ = \; & \limsup_{m \to +\infty} \limsup_{n \to +\infty} \mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0 | m_{\rho n} > 0) \\ = \; & \limsup_{m \to +\infty} \widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_m \neq 0). \end{aligned} \tag{21}$$

Step 2. By (15), we know that the sequence $(\widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_m \neq 0))_{m \geq 0}$ converges; its limit is given by the Lemma 3.2 below.

Lemma 3.2 *For any $\tilde{x} \in \mathbb{X}$ and $a > 0$,*

$$\lim_{m \to +\infty} \widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_m \neq 0) = \widehat{\mathbb{E}}_{\tilde{x},a} q^{(i)}_{\infty}. \tag{22}$$

Moreover, the following lemma deduces that the quantity $v(\tilde{x}, a) := \widehat{\mathbb{E}}_{\tilde{x},a} q^{(i)}_{\infty}$ does not vanish as m tends to ∞.

Lemma 3.3 *For any $\tilde{x} \in \mathbb{X}$ and $a > 0$,* $\quad \widehat{\mathbb{E}}_{\tilde{x},a} \sum\limits_{n=0}^{+\infty} e^{-S_n} < +\infty.$

Since $\tilde{\gamma}_k$ are bounded and $Y_{k+1,\infty} \in \mathcal{C} \cap \mathbb{S}^{p-1}$, by using Lemma 2.2 property 2), that is to say $|\tilde{x} R_n| \asymp \tilde{e}_i R_n Y_{n+1,\infty}$, Lemma 3.3 implies $\widehat{\mathbb{E}}_{\tilde{x},a} q^{(i)}_{\infty} < +\infty$ and $\widehat{\mathbb{E}}_{\tilde{x},a}(\sum\limits_{n=0}^{+\infty} e^{-S_n})^{-1} > 0$, which yields $0 < v(\tilde{x}, a) < +\infty$ for any $a > 0$.

Step 3. For $\rho > 1$ fixed, we decompose $\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n > 0)$ as $P_1(\rho, n) + P_2(\rho, n)$ with

$$P_1(\rho, n) := \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n > 0) - \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_{\rho n} > 0)$$

and

$$P_2(\rho, n) := \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_{\rho n} > 0).$$

Next, we get rid of ρ. Theorem 2.3 states that, as $n \to +\infty$

$$\mathbb{P}_{\tilde{x},a}(m_{\rho n} > 0) = \mathbb{P}_{\tilde{x},a}(\tau > \rho n) = \mathbf{m}_{\rho n}(\tilde{x}, a) \sim c_1 h(\tilde{x}, a) \frac{1}{\sqrt{\rho n}}$$

so that on one hand

$$\begin{aligned}
P_1(\rho,n) &= \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,\tau>n)-\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,\tau>\rho n)\\
&= \mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,n<\tau\leq\rho n)\\
&\leq \mathbb{P}_{\tilde{x},a}(n<\tau\leq\rho n)\\
&= \mathbb{P}_{\tilde{x},a}(\tau>n)-\mathbb{P}_{\tilde{x}}(\tau>\rho n)\\
&\sim c_1\frac{h(\tilde{x},a)}{\sqrt{n}}(1-\frac{1}{\sqrt{\rho}})\quad\text{as}\quad n\to+\infty \qquad (23)
\end{aligned}$$

and on the other hand, from (21) and (22),

$$P_2(\rho,n)=\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0|\tau>\rho n)\mathbb{P}_{\tilde{x},a}(\tau>\rho n)\sim c_1h(\tilde{x},a)v(\tilde{x},a)\frac{1}{\sqrt{\rho n}}\quad\text{as}\quad n\to+\infty. \qquad (24)$$

Hence, (23) and (24) yields

$$\begin{aligned}
\lim_{n\to+\infty}\sqrt{n}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,m_n>0) &= \lim_{n\to+\infty}\sqrt{n}P_1(\rho,n)+\lim_{n\to+\infty}\sqrt{n}P_2(\rho,n)\\
&= c_1h(\tilde{x},a)(1-\frac{1}{\sqrt{\rho}})+\frac{c_1}{\sqrt{\rho}}h(\tilde{x},a)v(\tilde{x},a).
\end{aligned}$$

The factor $(1-\frac{1}{\sqrt{\rho}})$ in (23) can be made arbitrary small by choosing ρ sufficiently closed to 1. Thus

$$\lim_{n\to+\infty}\sqrt{n}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,m_n>0) = c_1h(\tilde{x},a)v(\tilde{x},a). \qquad (25)$$

Step 4. For any $a>0$, we may decompose $\mathbb{P}^{(i)}(Z_n\neq 0)$ as

$$\mathbb{P}^{(i)}(Z_n\neq 0)=\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,m_n>0)+\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,m_n\leq 0). \qquad (26)$$

The first term of the right side of (26) is controled by (25). For the second term, we write

$$\begin{aligned}
\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0,m_n\leq 0) &= \mathbb{E}_{\tilde{x},a}\left[\mathbb{E}\left[Z_n\neq 0|f_0^{(i)},...,f_{n-1}\right];m_n\leq 0\right]\\
&= \mathbb{E}_{\tilde{x},a}\left[q_n^{(i)};m_n\leq 0\right].
\end{aligned}$$

Now, it is reasonable to control the quantity $q_n^{(i)}$; using Lemma 2.2, one gets

$$\begin{aligned}
(q_n^{(i)})^{-1} &= \frac{1}{\tilde{e}_iR_n\mathbf{1}}+\sum_{k=0}^{n-1}\frac{\gamma_k}{\tilde{e}_iR_kY_{k+1,n}}\\
&\geq \max_{0\leq k\leq n-1}\left\{\frac{\gamma_k}{\tilde{e}_iR_kY_{k+1,n}}\right\}\asymp\max_{0\leq k\leq n-1}\left\{\frac{1}{|\tilde{x}R_k|}\right\}\\
&\geq \frac{1}{\exp\left\{\min\limits_{0\leq k\leq n-1}(a+\ln|\tilde{x}R_k|)\right\}}.
\end{aligned}$$

Hence $q_n^{(i)} \preceq \exp(m_n(\tilde{x}, a))$ and by applying Theorem 2.3 equation (10), the second term of the right side of (26) becomes:

$$\begin{aligned}
\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n \leq 0) &= \mathbb{P}^{(i)}_{\tilde{x}}(Z_n \neq 0, m_n \leq -a) \\
&\preceq \mathbb{E}_{\tilde{x}}[\exp(m_n); m_n \leq -a] \\
&\leq \sum_{k=a}^{+\infty} e^{-k}\mathbb{P}_{\tilde{x}}(-k < m_n \leq -k+1) \\
&\leq \sum_{k=a}^{+\infty} e^{-k}\mathbb{P}_{\tilde{x},k}(\tau > n) \\
&\preceq \frac{1}{\sqrt{n}} \sum_{k=a}^{+\infty} (k+1)e^{-k}. \qquad (27)
\end{aligned}$$

Notice that the sum $\sum_{k=a}^{+\infty} (k+1)e^{-k}$ becomes arbitrarily small for sufficiently great a. Hence the quantity $\limsup_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n \leq 0)$ is over approximated by the same manner. On one hand,

$$c_1 h(\tilde{x}, a)v(\tilde{x}, a) = \lim_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n > 0) \leq \lim_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}(Z_n \neq 0).$$

On the other hand, by (25), (26) and (27), we have for some constant $c > 0$,

$$\begin{aligned}
c_1 h(\tilde{x}, a)v(\tilde{x}, a) &= \lim_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n > 0) \\
&= \lim_{n\to+\infty} [\sqrt{n}\mathbb{P}^{(i)}(Z_n \neq 0) - \sqrt{n}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n \neq 0, m_n \leq 0)] \\
&\geq \limsup_{n\to+\infty} [\sqrt{n}\mathbb{P}^{(i)}(Z_n \neq 0) - c\sum_{k=a}^{+\infty}(k+1)e^{-k}],
\end{aligned}$$

which implies

$$c_1 h(\tilde{x}, a)v(\tilde{x}, a) + c\sum_{k=a}^{+\infty}(k+1)e^{-k}] \geq \limsup_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}(Z_n \neq 0).$$

Since $\sum_{k=0}^{+\infty}(k+1)e^{-k} < +\infty$, for any $\varepsilon > 0$, we can always choose a to be great enough so that $c\sum_{k=a}^{+\infty}(k+1)e^{-k} < \varepsilon$. Hence, for any $\varepsilon > 0$,

$$\begin{aligned}
c_1 h(\tilde{x}, a)v(\tilde{x}, a) &\leq \liminf_{n\to+\infty} \sqrt{n}\mathbb{P}(Z_n \neq 0) \\
&\leq \limsup_{n\to+\infty} \sqrt{n}\mathbb{P}(Z_n \neq 0) \leq c_1 h(\tilde{x}, a)v(\tilde{x}, a) + \varepsilon \qquad (28)
\end{aligned}$$

if only a is chosen great enough. Remind that $v(\tilde{x}, a) > 0$ for any $a > 0$ and $h(\tilde{x}, a) > 0$ for a large enough; by (25) the quantity $h(\tilde{x}, a)v(\tilde{x}, a)$ is increasing in a, hence $\beta :=$

$c_1 \lim_{a\to+\infty} h(\tilde{x},a)v(\tilde{x},a)$ is strictly positive. For any $\varepsilon > 0$, by (25), it follows that

$$\begin{aligned} 0<\beta &= \lim_{a\to+\infty} c_1 h(\tilde{x},a)v(\tilde{x},a) \\ &= \lim_{a\to+\infty}\lim_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}_{\tilde{x},a}(Z_n\neq 0, m_n>0) \\ &\le \lim_{a\to+\infty}\limsup_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}(Z_n\neq 0) \\ &\le \limsup_{n\to+\infty} \sqrt{n}\mathbb{P}^{(i)}(Z_n\neq 0) \\ &\le c_1 h(\tilde{x},a)v(\tilde{x},a)+\varepsilon<+\infty. \end{aligned} \tag{29}$$

Therefore, from (28) and (29), the assertion of the theorem arrives.

3.4 Proof of Theorem 1.2

First, for any $n \geq 1$ and $s = (s_1, \ldots, s_p)$, we denote $F_n(s) = f_0(f_1(\ldots(f_{n-1}(s))\ldots))$. By definition of $q_n^{(i)}$, we have for any $0 \leq m < n$,

$$q_n^{(i)} = \tilde{e}_i(F_m(\mathbf{1}) - F_m(z)),$$

where $z = z(m,n) = f_m(\ldots(f_{n-1}(0))\ldots)$. The Mean Value Theorem yields

$$\begin{aligned} \tilde{e}_i(F_m(\mathbf{1}) - F_m(z)) &\le \sum_{j=1}^{p}\left(\int_0^1 \frac{\partial F_m}{\partial s_j}(z+(1-z)t)dt\right)(1-z_j) \\ &\le \sum_{j=1}^{p} \frac{\partial F_m}{\partial s_j}(\mathbf{1}) \\ &= \tilde{e}_i M_0\ldots M_{m-1}\mathbf{1}. \end{aligned}$$

Therefore, using Lemma 2.2, we have for any $0 \leq m \leq n$ and $x \in \mathbb{X}$,

$$q_n^{(i)} \le \tilde{e}_i M_0\ldots M_{m-1}\mathbf{1} \asymp |\tilde{x}R_m| = \exp(S_m(\tilde{x},0)),$$

which yields $q_n^{(i)} \preceq \exp(m_n(\tilde{x},0))$ and

$$\mathbb{E}[q_n^{(i)}] \ \preceq\ \mathbb{E}[e^{m_n(\tilde{x},0)}] \ =\ \mathbb{E}_{\tilde{x}}[e^{m_n}].$$

Using the same trick like in (27), we can deduce that there exists a constant c_2 such that

$$\mathbb{E}_{\tilde{x}}[e^{m_n}] = \mathbb{E}_{\tilde{x}}[e^{m_n}; m_n\le 0] \sim \frac{c_2}{\sqrt{n}},$$

and thus the upper estimate in equation (3) arrives.

To obtain the lower estimate in (3), for any $\mathbb{R}$-valued multi-dimensional generating function $f(s)$, $s = (s_1, \ldots, s_p)^T$, we obtain (see for instance formulas (64) and (65) in [12])

$$f(s) \le 1 - \left(\sum_{i=1}^{p}\frac{\partial f}{\partial s_i}(\mathbf{1})(1-s_i)\right)\left(1+\frac{\sum_{i,j=1}^{p}\frac{\partial^2 f}{\partial s_i\partial s_j}(\mathbf{1})(1-s_j)(1-s_i)}{\sum_{l=1}^{p}\frac{\partial f}{\partial s_l}(\mathbf{1})(1-s_l)}\right)^{-1}. \tag{30}$$

We set $g_\xi(s) = \mathbf{1} - \dfrac{M_\xi(\mathbf{1}-s)}{1+\widetilde{\Gamma}_\xi(\mathbf{1}-s)}$, where M_ξ is the mean matrix of $f_\xi(s)$ and $\widetilde{\Gamma}_\xi = (A,\dots,A)$. Denote $g_{\xi_n}(s) = g_n(s)$. Applying inequality (30) with $f = f_\xi^{(i)}$, we may write

$$f_\xi^{(i)}(s) \le g_\xi^{(i)}(s), \quad i = 1,\dots,p,$$

which yields

$$\mathbb{E}[\mathbf{1} - g_0(g_1(\dots(g_{n-1}(0))\dots))] \le \mathbb{E}[\mathbf{1} - f_0(f_1(\dots(f_{n-1}(0))\dots))]. \tag{31}$$

The lower estimate in equation (1.2) appears by applying Theorem 1.1 to the left side of equation (31). Therefore, the assertion of the Theorem 1.2 arrives.

4 Proof of facts

We first give some hints for the proof of Lemma 2.2. Lemmas 3.1, 3.2 and 3.3 are listed in the order of our use; since they are dependent, we first prove Lemma 3.3, then Lemma 3.2 and at last Lemma 3.1.

4.1 Proof of Lemma 2.2

First, we obtain (32) by formally using (4)

$$|g| = \sum_{i,j=1}^{p} g(i,j) \overset{p^2B^2}{\asymp} g(k,l). \tag{32}$$

Further properties can be easily deduced from (32). Indeed, the assertions we need are obvious by noticing that

$$|gx| = \sum_{i,j=1}^{p} g(i,j)x_j \overset{p^3B^2}{\asymp} |g|,$$

$$\tilde{y}gx = \sum_{i,j=1}^{p} y_i g(i,j)x_j \overset{p^2B^2}{\asymp} |g|,$$

$$|gh| = \sum_{i,j,k=1}^{p} g(i,j)h(j,k) \overset{p^7B^4}{\asymp} |g||h|.$$

4.2 Proof of Lemma 3.3

Before going into the proof, we first claim that in the critical case, for any $\delta > 0$ and c given from Lemma 2.2, there exists $\kappa = \kappa(\delta, c) \ge 1$ such that

$$\mu^{*\kappa}(E_\delta) := \mu^{*\kappa}\{g : \forall \tilde{x} \in \mathbb{X},\ \ln|\tilde{x}g| \ge \delta\} > 0. \tag{33}$$

Indeed, let $\tau' := \inf\{n \ge 1 : \ln|R_n| \ge \ln c + \delta\}$; the random variable τ' is a stopping time with respect to the natural filtration $(\sigma(M_0,\dots,M_k))_{k\ge0}$ and $\mathbb{P}$-a.s. finite since $\limsup_{n\to+\infty} \ln|R_n| = +\infty$.

Therefore, for any $\delta > 0$ and c given from Lemma 2.2, there exists $\kappa \geq 1$ such that $\mathbb{P}(\tau' = \kappa) = p > 0$. Moreover, we also have

$$\begin{aligned}
\mathbb{P}(\ln |R_\kappa| \geq \ln c + \delta) &\geq \mathbb{P}(\ln |R_\kappa| \geq \ln c + \delta, \tau' = \kappa) \\
&= \mathbb{P}(\ln |R_{\tau'}| \geq \ln c + \delta, \tau' = \kappa) \\
&= \mathbb{P}(\tau' = \kappa) = p > 0.
\end{aligned}$$

Since for any $\tilde{x} \in \mathbb{X}$, $g \in G$, $|gx| \geq \frac{|g|}{c}$, it follows that

$$\{g : \ln |g| \geq \ln c + \delta\} \subset \{g : \forall \tilde{x} \in \mathbb{X}, \ln |\tilde{x}g| \geq \delta\}.$$

Thus,

$$0 < \mathbb{P}(\ln |R_\kappa| \geq \ln c + \delta) = \mu^{*\kappa}\{g : \ln |g| \geq \ln c + \delta\} \leq \mu^{*\kappa}\{g : \forall \tilde{x} \in \mathbb{X},\ \ln |\tilde{x}g| \geq \delta\},$$

which is the assertion of the claim (33).

Now, let us go into the proof of Lemma 3.3. For any $\tilde{x} \in \mathbb{X}$, $a > 0$ and $\lambda \in (0, 1)$, there exists some constant $C(\lambda) > 0$ such that $(t + 1)e^{-t} \leq C(\lambda)e^{-\lambda t}$ for any $t > 0$ and c is introduced in equation (8). Hence

$$\begin{aligned}
\widehat{\mathbb{E}}_{\tilde{x},a}\Big[\sum_{n=0}^{+\infty} e^{-S_n}\Big] &\leq 1 + \frac{1}{h(\tilde{x}, a)} \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x},a}\left[e^{-S_n} h(X_n, S_n); S_0 > 0, \dots, S_n > 0\right] \\
&\leq 1 + \frac{c}{h(\tilde{x}, a)} \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x},a}\left[e^{-S_n}(1 + S_n); S_0 > 0, \dots, S_n > 0\right] \\
&\leq 1 + \frac{cC(\lambda)}{h(\tilde{x}, a)} \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x},a}\Big[e^{-\lambda S_n}; S_0 > 0, \dots, S_n > 0\Big] \\
&\leq 1 + \frac{cC(\lambda)}{h(\tilde{x}, a)} \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x},a}\Big[e^{-\lambda S_n}; S_1 > 0, \dots, S_n > 0\Big]
\end{aligned}$$

Now, we define a function Φ for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$ as follow:

$$\Phi(\tilde{x}, a) := \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x},a}\Big[e^{-\lambda S_n}; S_1 > 0, \dots, S_n > 0\Big].$$

Notice that $S_0 := a$ with respect to $\mathbb{E}_{\tilde{x},a}$ for any $\tilde{x} \in \mathbb{X}$, we may skip the event $[S_0 > 0]$ for any positive a. This is a trick to deal with our problem since $\Phi(\tilde{x}, a) = 0$ whenever $a \leq 0$ and we can not do anything more. Hence, it suffices to prove for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$,

$$\Phi(\tilde{x}, a) < +\infty, \tag{34}$$

and the assertion of Lemma 3.3 arrives for $a > 0$.

Notice that for any $\tilde{x} \in \mathbb{X}$, the function $\Phi(\tilde{x}, .)$ increases on $\mathbb{R}$. We take into account the spirit of the strategy of the proof of Lemma 3.2 in [5]. In the multi-dimensional case, it is more complicated to apply the duality principle, namely $\mathcal{L}(M_0, M_1, ..., M_n) = \mathcal{L}(M_n, ..., M_1, M_0)$, and we can only prove that for some $a_0 < 0$, the quantity $\Phi(\tilde{x}, a_0)$ is finite. Unfortunately, $\Phi(\tilde{x}, a_0)$ may vanish and then we can not say anything else about $\Phi(\tilde{x}, a)$ for $a > a_0$. To avoid this difficulty, we skip the first κ steps by introducing the functions Φ_κ associated with the κ^{th} power of convolution $\mu^{*\kappa}$ of μ. For any $\tilde{x} \in \mathbb{X}, a \in \mathbb{R}$, let

$$\Phi_\kappa(\tilde{x}, a) := \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x}} \Big[e^{-\lambda S_{n\kappa}}; S_\kappa > 0, \dots, S_{n\kappa} > 0 \Big].$$

The relation is that $\Phi(\tilde{x}, a) \preceq \Phi_\kappa(\tilde{x}, a)$ for any $\tilde{x} \in \mathbb{X}, a \in \mathbb{R}$. Then, by using the duality principle, we bound from above $\Phi_\kappa(\tilde{x}, a)$ by a new quantity $\Psi_\kappa(\tilde{x})$ defined below for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$. Finally, we prove $\Psi_\kappa(\tilde{x}) < +\infty$ by using the ascending ladder epochs associated to the Markov walk $(L_n \cdot x, \ln |L_n x|)_{n \geq 0}$ and the Elementary Renewal Theorem.

We set $L_0 = 0$ and denote $L_n := M_{n-1} \dots M_0$ the left product of the matrices $M_0, \dots, M_n$ when $n \geq 1$. For any $\tilde{x} \in \mathbb{X}, a \in \mathbb{R}$, let

$$\Psi_\kappa(\tilde{x}) := \sum_{n=1}^{+\infty} \mathbb{E} \Big[|L_{n\kappa} x|^{-\lambda}; |L_{n\kappa} x| > |L_{(n-1)\kappa} x|, \dots, |L_{n\kappa} x| > 1 \Big].$$

Property (34) is a direct consequence of the four steps following:

1. For any $\kappa \geq 1$, there exists $C(\kappa) > 0$ such that, for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$,
$$\Phi(\tilde{x}, a) \leq C(\kappa)(1 + \Phi_\kappa(\tilde{x}, a)).$$

2. If there exist some $\kappa \geq 1, \tilde{x}_0 \in \mathbb{X}$ and $a_0 < 0$ such that $0 < \Phi_\kappa(\tilde{x}_0, a_0) < +\infty$, then
$$\forall \tilde{x} \in \mathbb{X}, \forall a \in \mathbb{R} \qquad \Phi_\kappa(\tilde{x}, a) < +\infty.$$

3. There exist $C_1 > 0$ and $a_1 < 0$ such that for any $\kappa \geq 1$, $\tilde{x} \in \mathbb{X}$ and $a < a_1$
$$\Phi_\kappa(\tilde{x}, a) \overset{C_1}{\preceq} \Psi_\kappa(\tilde{x}).$$

4. For any $\kappa \geq 1$ and $\tilde{x} \in \mathbb{X}$
$$\Psi_\kappa(\tilde{x}) < +\infty.$$

Roughly speaking, on one hand, for any $a_0 \leq a_1 < 0$, we can always choose some δ_0 such that $\delta_0 > -a_0 > 0$. For each δ_0, there exists $\kappa_0 \geq 1$ such that $\mathbb{P}(\ln |\tilde{x} R_{\kappa_0}| \geq \delta_0) > 0$ (see (33) above). Since $\delta_0 > -a_0$, we have $\mathbb{P}_{\tilde{x},a}(S_{\kappa_0} > 0) > 0$, which implies $\Phi_{\kappa_o}(\tilde{x}_0, a_0) > 0$. On the other hand, since $a_0 \leq a_1$, step 3 and step 4 yield $\Phi_{\kappa_o}(\tilde{x}_0, a_0) < +\infty$. Therefore, we can apply step 2 and it yields $\Phi_\kappa(\tilde{x}, a) < +\infty$ for any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$. Finally, thanks to Step 1, (34) arrives.

Step 1. It is easy to see that

$$\begin{aligned}
\Phi(\tilde{x}, a) &\leq \sum_{r=1}^{\kappa-1} \mathbb{E}_{\tilde{x}}[e^{-\lambda S_r}] + \sum_{n=1}^{+\infty} \sum_{r=0}^{\kappa-1} \mathbb{E}_{\tilde{x},a}[e^{-\lambda S_{n\kappa+r}}; S_\kappa > 0, \dots, S_{n\kappa} > 0] \\
&\leq \sum_{r=1}^{\kappa-1} \mathbb{E}_{\tilde{x}} \Big[e^{-\lambda S_r} \Big] + \sum_{n=1}^{+\infty} \mathbb{E}_{\tilde{x},a} \Big[e^{-\lambda S_{n\kappa}}; S_\kappa > 0, \dots, S_{n\kappa} > 0 \Big] \times \sum_{r=0}^{\kappa-1} \sup_{\tilde{y} \in \mathbb{X}} \mathbb{E}_{\tilde{y},a} \Big[e^{-\lambda S_r} \Big] \\
&\leq \Big(\sum_{r=0}^{\kappa-1} \sup_{\tilde{y} \in \mathbb{X}} \mathbb{E}_{\tilde{y},a} \Big[e^{-\lambda S_r} \Big] \Big) (1 + \Phi_\kappa(\tilde{x}, a)),
\end{aligned}$$

which yields to the expected result with $0 < C(\kappa) = \sum_{r=0}^{\kappa-1} \sup_{\tilde{y} \in \mathbb{X}} \mathbb{E}_{\tilde{y},a}[e^{-\lambda S_r}] < +\infty$.

Step 2. The inequality $\Phi_\kappa(\tilde{x}_0, a_0) > 0$ implies that $\mathbb{P}(\ln|\tilde{x}_0 R_\kappa| > -a_0) > 0$; we thus fix $\delta > -a_0 > 0$ and $\kappa \geq 1$ such that $\mu^{*\kappa}(E_\delta) > 0$. Since $a_0 < 0$, this property may hold only when κ is large enough; this happens for instance when the support of μ is bounded.
To simplify the notations, we assume that $-a_0 < \delta$ where δ is given by H5. We set $\kappa = 1$ and write

$$\begin{aligned}
\Phi(\tilde{x}_0, a_0) &= \sum_{n=1}^{+\infty} \mathbb{E}[|\tilde{x}_0 R_n|^{-\lambda}; |\tilde{x}_0 R_1| > e^{-a_0}, \ldots, |\tilde{x}_0 R_n| > e^{-a_0}] \\
&\geq \int_{\{g \in G : |\tilde{x}_0 g| \geq e^{-a_0}\}} \sum_{n=2}^{+\infty} \mathbb{E}\left[|\tilde{x}_0 g R_{1,n}|^{-\lambda}; |\tilde{x}_0 g| > e^{-a_0}, \ldots, |\tilde{x}_0 g R_{1,n}| > e^{-a_0}\right] \mu(dg) \\
&\geq \int_{E_\delta} \sum_{n=2}^{+\infty} \mathbb{E}[|\tilde{x}_0 g R_{1,n}|^{-\lambda}; \\
&\qquad\qquad |\tilde{x}_0 g| \geq e^{\delta} > e^{-a_0}, |\tilde{x}_0 g R_{1,2}| > e^{-a_0}, \ldots, |\tilde{x}_0 g R_{1,n}| > e^{-a_0}]\mu(dg) \\
&= \int_{E_\delta} |\tilde{x}_0 g|^{-\lambda} \sum_{m=1}^{+\infty} \mathbb{E}[|(\tilde{x}_0 \cdot g) R_m|^{-\lambda}; \\
&\qquad\qquad |(\tilde{x}_0 \cdot g) R_1| > e^{-a_0 - \ln|\tilde{x}_0 g|}, \ldots, |(\tilde{x}_0 \cdot g) R_m| > e^{-a_0 - \ln|\tilde{x}_0 g|}]\mu(dg) \\
&= \int_{E_\delta} |\tilde{x}_0 g|^{-\lambda} \Phi(\tilde{x}_0 \cdot g, a_0 + \ln|\tilde{x}_0 g|)\mu(dg) \\
&\geq \int_{E_\delta} |\tilde{x}_0 g|^{-\lambda} \Phi(\tilde{x}_0 \cdot g, a_0 + \delta)\mu(dg).
\end{aligned}$$

Consequently, if $\Phi(\tilde{x}_0, a_0) < +\infty$ then $\Phi(\tilde{x}_0 \cdot g, a_0 + \delta) < +\infty$ for μ-almost all $g \in E_\delta$ and by iterating this argument, there thus exists a sequence $(g_k)_{k \geq 1}$ of elements of E_δ such that

$$\forall k \geq 1, \quad \Phi(\tilde{x}_0 \cdot g_1 \cdots g_k, a_0 + k\delta) < +\infty.$$

By Lemma 2.2, for any $\tilde{x}, \tilde{y} \in \mathbb{X}$ and $a \in \mathbb{R}$

$$\Phi(\tilde{x}, a - \ln c) \leq c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}[|R_n|^{-\lambda}; |R_1| > e^{-a}, \ldots, |R_n| > e^{-a}] \leq c^{2\lambda} \Phi(\tilde{y}, a + \ln c);$$

it follows that, by choosing k sufficiently great such that $a_0 + k\delta > a + 2\ln c$, we have

$$\Phi(\tilde{x}, a) \leq \Phi(\tilde{x}_0 \cdot g_1 \cdots g_k, a + 2\ln c) \leq \Phi(\tilde{x}_0 \cdot g_1 \cdots g_k, a_0 + k\delta) < +\infty.$$

Step 3. For any $0 \leq k < n$, denote $L_{n,k} := M_{n-1} \ldots M_k$ and $L_{n,k} = I$ otherwise. Let $c > 1$ be the constant given by Lemma 2.2. For any $\tilde{x} \in \mathbb{X}$ and $a \in \mathbb{R}$, by using Lemma 2.2, we may write

$$\begin{aligned}
\Phi_\kappa(\tilde{x}, a) &= \sum_{n=1}^{+\infty} \mathbb{E}\left[|\tilde{x} R_{n\kappa}|^{-\lambda}; |\tilde{x} R_\kappa| > e^{-a}, \ldots, |\tilde{x} R_{n\kappa}| > e^{-a}\right] \\
&\leq c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}\left[|R_{n\kappa}|^{-\lambda}; |R_\kappa| > \frac{e^{-a}}{c}, \ldots, |R_{n\kappa}| > \frac{e^{-a}}{c}\right]
\end{aligned}$$

so that, by duality principle and Lemma 2.2,

$$
\begin{aligned}
\Phi_\kappa(\tilde{x},a) &\le c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}\left[|L_{n\kappa}|^{-\lambda}; \left|L_{n\kappa,(n-1)\kappa}\right| > \frac{e^{-a}}{c}, \ldots, |L_{n\kappa}| > \frac{e^{-a}}{c}\right] \\
&= c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}[|L_{n\kappa}|^{-\lambda}; \left|L_{n\kappa,(n-1)\kappa}\right| \times \left|L_{(n-1)\kappa}\right| > \left|L_{(n-1)\kappa}\right| \frac{e^{-a}}{c}, \\
&\qquad\qquad \ldots, |L_{n\kappa}| > \frac{e^{-a}}{c}] \\
&\le c^\lambda \sum_{n=1}^{+\infty} \mathbb{E}\left[|L_{n\kappa}|^{-\lambda}; |L_{n\kappa}| > \left|L_{(n-1)\kappa}\right| \frac{e^{-a}}{c^2}, \ldots, |L_{n\kappa}| > \frac{e^{-a}}{c^2}\right] \\
&\le c^{2\lambda} \sum_{n=1}^{+\infty} \mathbb{E}\left[|L_{n\kappa}x|^{-\lambda}; |L_{n\kappa}x| > \left|L_{(n-1)\kappa}x\right| \frac{e^{-a}}{c^4}, \ldots, |L_{n\kappa}x| > \frac{e^{-a}}{c^4}\right]
\end{aligned}
$$

Consequently, setting $a_1 := -4\ln c$ and using the fact that the map $a \mapsto \Phi_\kappa(\tilde{x},a)$ is non decreasing for any $a \in \mathbb{R}$, one may write $\Phi_\kappa(\tilde{x},a) \le \Psi(\tilde{x})$ as long as $a < a_1$.

Step 4. To simplify the notations, we assume here $\kappa = 1$; the proof is the same when $\kappa \ge 2$. For any $\tilde{x} \in \mathbb{X}$ and $n \ge 0$, set $X'_n := L_n \cdot x$ and $S'_n := \ln|L_n x|$; the random process $(X'_n, S'_n)_{n\ge 0}$ is a Markov walk on $\mathbb{X} \times \mathbb{R}$ starting from $(x, 0)$ and whose transitions are governed by the ones of the Markov chain $(X'_n)_{n\ge 0}$ on $\mathbb{X}$. To study the quantity $\Psi(\tilde{x})$, we follow the strategy developed in the case of one dimensional random walks on $\mathbb{R}$ with independent increments and we thus introduce the sequence $(\eta_j)_{j\ge 0}$ of ladder epochs of $(S'_n)_n$ defined by

$$\eta_1 = 0,\ \eta_{j+1} = \eta_{j+1}(x) := \min\left\{n > \eta_j : \ln|L_n x| > \ln\left|L_{\eta_j} x\right|\right\},\ j \ge 0.$$

For any $\tilde{x} \in \mathbb{X}$, one may write

$$
\begin{aligned}
\Psi(\tilde{x}) &= \sum_{n=1}^{+\infty} \mathbb{E}\left[|L_n x|^{-\lambda}; \exists j \ge 1 : n = \eta_j\right] \\
&= \sum_{j=1}^{+\infty} \mathbb{E}\left[\left|L_{\eta_j} x\right|^{-\lambda}\right]. \qquad (35)
\end{aligned}
$$

Let Q' denote the transition kernel of the Markov walk $(X'_n, S'_n)_n$ and $G_{Q'} := \sum_{n=0}^{+\infty} Q'^n$ its Green kernel. The sub-process $(X'_{\eta_j}, S'_{\eta_j})_{j\le 0}$ is also a Markov chain, its transition kernel Q'_η is given by: for any bounded Borel function $\phi : \mathbb{X} \times \mathbb{R} \to \mathbb{C}$ and for any $x \in \mathbb{X}$, $a \in \mathbb{R}$,

$$
\begin{aligned}
Q'_\eta \phi(x,a) &= \mathbb{E}\left[\phi(X'_{\eta_1}, a + S'_{\eta_1}) | X'_0 = x\right] \\
&= \sum_{n=1}^{+\infty} \mathbb{E}\left[\phi(L_n \cdot x, a + \ln|L_n x|); \eta_1 = n\right] \\
&= \sum_{n=1}^{+\infty} \mathbb{E}[\phi(L_n \cdot x, a + \ln|L_n x|); \\
&\qquad\qquad |L_1 x| \le 1, \cdots, |L_{n-1} x| \le 1, |L_n x| > 1].
\end{aligned}
$$

Let G'_η denote the Green kernel associated with the process $(X'_{\eta_j}, S'_{\eta_j})_{j\geq 0}$; by (35)

$$
\begin{aligned}
1+\Psi(\tilde{x}) &= \sum_{j=0}^{+\infty} \mathbb{E}\left[|L_{\eta_j}x|^{-\lambda}\right] \\
&= \sum_{j=0}^{+\infty} \int_{\mathbb{X}}\int_{\mathbb{R}} e^{-\lambda a}(Q'_\eta)^j((x,0),dyda) \\
&= \int_{\mathbb{X}}\int_{\mathbb{R}} e^{-\lambda a} G'_\eta((x,0),dyda).
\end{aligned}
$$

The Markov walk $(X'_n, S'_n)_{n\geq 0}$ has been studied by many people (see for instance [3], [8] or [6]). All the work are based on the fact that the transition kernel of the chain $(X'_n)_n$ has some "nice" spectral properties, namely its restriction to the space of Lipschitz functions on $\mathbb{X}$ is quasi-compact. In particular, it allows these authors to prove that the classical renewal theorem remains valid for this Markov walk on $\mathbb{X}\times\mathbb{R}$ as long as it is not centered, that is $\pi = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}[\ln|L_n|] \neq 0$; in this case one may prove in particular that, for any $\tilde{x}\in\mathbb{X}$, the quantity $G_{Q'}((x,0),\mathbb{X}\times[0,a])$ is equivalent to $\frac{a}{\pi}$ as $a\to+\infty$ [8]. For the behavior as $a\to+\infty$ of $G'_\eta((x,0),\mathbb{X}\times[0,a])$, the situation is way different. On one hand, it is easier since for any $j\geq 1$ the random variables S'_{η_j} are strictly positive, one might thus expect a similar result; on the other hand, the control of the spectrum of the transition kernel Q'_η remains unfortunately unknown in this circumstance, in particular the transition kernel Q'_η does not even act on the space of continuous functions on $\mathbb{X}$!

Nevertheless, we have the following weak result with the postponed proof at the end of this subsection.

Fact 4.1 *There exists $C>0$ such that for any $\tilde{x}\in\mathbb{X}$ and $a>0$*

$$
G'_\eta((x,0),\mathbb{X}\times[0,a]) = \sum_{j=0}^{+\infty}\mathbb{P}([\ln|L_{\eta_j}x| \leq a) \leq Ca.
$$

It follows that

$$
\begin{aligned}
1+\Psi(\tilde{x}) &= \int_{\mathbb{X}}\int_{\mathbb{R}^+_*} e^{-\lambda a} G'_\eta((x,0),dyda) \\
&\leq e^{\lambda}\sum_{a=1}^{+\infty} e^{-\lambda a} G'_\eta((x,0),\mathbb{X}\times[a-1,a]) \\
&\leq e^{\lambda}\sum_{a=1}^{+\infty} e^{-\lambda a} G'_\eta((x,0),\mathbb{X}\times[0,a]) \\
&\leq Ce^{\lambda}\sum_{a=1}^{+\infty} a e^{-\lambda a} < +\infty.
\end{aligned}
$$

To complete the proof of Step 4, it remains to prove Fact 4.1. First, by definition of E_δ, for any $j\geq 0$ and $\tilde{x}\in\mathbb{X}$, we may write $S'_{\eta_{j+1}} - S'_{\eta_j} \geq \delta 1_{E_\delta}(M_{\eta_j})$; setting $\varepsilon_j := 1_{E_\delta}(M_{\eta_j})$, this yields $S'_{\eta_j} \geq \delta(\varepsilon_0 + \ldots + \varepsilon_{j-1})$ so that

$$\begin{aligned}
G'_\eta((x,0), \mathbb{X} \times [0,a]) &= \sum_{j=0}^{+\infty} \mathbb{P}(X'_{\eta_j} \in \mathbb{X}, S'_{\eta_j} \in [0,a] | X'_0 = x) \\
&\leq \sum_{j=0}^{+\infty} \mathbb{E}\left[1_{[0,a]}(S'_{\eta_j}) | X'_0 = x\right] \\
&\leq \sum_{j=0}^{+\infty} \mathbb{E}\left[1_{[0,a]}(\delta(\varepsilon_0 + \cdots \varepsilon_{j-1}))\right].
\end{aligned}$$

To conclude, we use the fact that $(\varepsilon_i)_{i\geq 0}$ is a sequence of i.i.d. random variables; the Elementary Renewal Theorem for the Bernoulli random walk process $[(\varepsilon_0 + \ldots + \varepsilon_{j-1})]_{j\geq 0}$ implies

$$G'_\eta((x,0), X \times [0,a]) \leq \mathbb{E}\left[\sum_{j=1}^{+\infty} 1_{[0,a]}(\delta(\varepsilon_0 + \ldots + \varepsilon_{j-1}))\right] \preceq a.$$

To check that the ε_j are i.i.d., we set $E^0 = G \setminus E_\delta, E^1 = E_\delta$ and we fix $k \geq 1$ and $e_0, \ldots, e_k \in \{0,1\}$; since $[g_0 \in E^{e_0}, \ldots, g_{n_{k-1}} \in E^{e_{k-1}}, \eta_1 = n_1, \cdots, \eta_k = n_k]$ belong to $\sigma(g_0, \cdots, g_{n_{k-1}})$, a straightforward computation yields

$$\begin{aligned}
\mathbb{P}(\varepsilon_0 = e_0, \ldots, \varepsilon_k = e_k) &= \mathbb{P}(g_0 \in E^{e_0}, \ldots, g_{\eta_k} \in E^{e_k}) \\
&= \sum_{1 \leq n_1 < \ldots < n_k} \mathbb{P}(g_0 \in E^{e_0}, g_{n_1} \in E^{e_1}, \ldots, g_{n_k} \in E^{e_k}, \\
&\qquad\qquad \eta_1 = n_1, \cdots, \eta_k = n_k) \\
&= \sum_{1 \leq n_1 < \ldots < n_k} \mathbb{P}(g_0 \in E^{e_0}, g_{n_1} \in E^{e_1}, \ldots, g_{n_{k-1}} \in E^{e_{k-1}}, \\
&\qquad\qquad \eta_1 = n_1, \cdots, \eta_k = n_k) \times \mathbb{P}(g_{n_k} \in E^{e_k}) \\
&= \mathbb{P}(g_0 \in E^{e_0}, \ldots, g_{\eta_{k-1}} \in E^{e_{k-1}}) \mu(E^{e_k})
\end{aligned}$$

and the assertion arrives by induction.

4.3 Proof of Lemma 3.2

We claim that

$$\lim_{n\to+\infty} \widehat{\mathbb{E}}_{\tilde{x},a} \left| (q_n^{(i)})^{-1} - (q_\infty^{(i)})^{-1} \right| = 0. \tag{36}$$

By definition, the quantities $q_n^{(i)}$ are always less than or equal to 1. Therefore, (36) implies that the same property holds $\widehat{\mathbb{P}}_{\tilde{x},a}$-almost surely for $q_\infty^{(i)}$. Hence, $\left| q_n^{(i)} - q_\infty^{(i)} \right| = q_n^{(i)} q_\infty^{(i)} \left| (q_n^{(i)})^{-1} - (q_\infty^{(i)})^{-1} \right| \leq \left| (q_n^{(i)})^{-1} - (q_\infty^{(i)})^{-1} \right|$. Using (36) again, we find that $\lim_{n\to+\infty} \widehat{\mathbb{E}}_{\tilde{x},a} \left| q_n^{(i)} - q_\infty^{(i)} \right| = 0$. In particular,

$$\lim_{n\to+\infty} \widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_n \neq 0) = \lim_{n\to+\infty} \widehat{\mathbb{E}}_{\tilde{x},a} q_n^{(i)} = \widehat{\mathbb{E}}_{\tilde{x},a} q_\infty^{(i)},$$

which is the assertion of (22). Finally, it remains to verify (36). From (11) and (12), for any $0 \leq l \leq n$, it follows that

$$\left| (q_n^{(i)})^{-1} - (q_\infty^{(i)})^{-1} \right| \leq \left| \frac{1}{\tilde{e}_i R_n \mathbf{1}} \right| + \sum_{k=0}^{l-1} \left| \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,n}} - \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}} \right| + \sum_{k=l}^{+\infty} \left| \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}} \right|.$$

Taking expectations with respect to $\widehat{\mathbb{P}}_{\tilde{x},a}$, we obtain

$$\widehat{\mathbb{E}}_{\tilde{x},a}\left|(q_n^{(i)})^{-1}-(q_\infty^{(i)})^{-1}\right| \le \widehat{\mathbb{E}}_{\tilde{x},a}\left|\frac{1}{\tilde{e}_i R_n \mathbf{1}}\right| + \sum_{k=0}^{l-1}\widehat{\mathbb{E}}_{\tilde{x},a}\left|\frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,n}} - \frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}}\right| + \sum_{k=l}^{+\infty}\widehat{\mathbb{E}}_{\tilde{x},a}\left|\frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}}\right|.$$

Let us denote A_n, B_n and C_n repectively as below,

$$\begin{aligned}
A_n &= \widehat{\mathbb{E}}_{\tilde{x},a}\left|\frac{1}{\tilde{e}_i R_n \mathbf{1}}\right|,\\
B_n &= \sum_{k=0}^{l-1}\widehat{\mathbb{E}}_{\tilde{x},a}\left|\gamma_k\frac{\tilde{e}_i R_k (Y_{k+1,n}-Y_{k+1,\infty})}{(\tilde{e}_i R_k Y_{k+1,n})(\tilde{e}_i R_k Y_{k+1,\infty})}\right|,\\
C_l &= \sum_{k=l}^{+\infty}\widehat{\mathbb{E}}_{\tilde{x},a}\left|\frac{\gamma_k}{\tilde{e}_i R_k Y_{k+1,\infty}}\right|.
\end{aligned}$$

By using Lemma 2.2, it is obvious that Lemma 3.3 implies

$$\sum_{k=0}^{+\infty}\widehat{\mathbb{E}}_{\tilde{x},a}\left[|R_k|^{-1}\right] < +\infty. \tag{37}$$

Besides, it is also an immediate consequence of Lemma 2.2 that

$$A_n \preceq \widehat{\mathbb{E}}_{x,a}\frac{1}{|R_n|}, \tag{38}$$

$$\left|\gamma_k\frac{\tilde{e}_i R_k (Y_{k+1,n}-Y_{k+1,\infty})}{(\tilde{e}_i R_k Y_{k+1,n})(\tilde{e}_i R_k Y_{k+1,\infty})}\right| \preceq \frac{1}{|R_k|}, \tag{39}$$

$$C_n \preceq \sum_{k=l}^{+\infty}\widehat{\mathbb{E}}_{x,a}\frac{1}{|R_k|}. \tag{40}$$

Hence, (37) and (38) implies $A_n \to 0$ as $n \to \infty$. For B_n, using (37), (39) and the fact that $Y_{k,n} \to Y_{k,\infty}$ $\mathbb{P}$-almost surely, we may apply the Dominated Convergence Theorem. Thanks to (37) and (40), C_l can be made arbitrarily small by choosing l sufficiently great.

4.4 Proof of Lemma 3.1

Assume $0 \le m \le n$. Using first (18), (14), and then (9), we find that

$$\begin{aligned}
& \mathbb{P}_{\tilde{x},a}^{(i)}(Z_m \neq 0, Z_n = 0 | m_{\rho n} > 0)\\
= \;& \mathbb{P}_{\tilde{x},a}^{(i)}(Z_m \neq 0 | m_{\rho n} > 0) - \mathbb{P}_{\tilde{x},a}^{(i)}(Z_n \neq 0 | m_{\rho n} > 0)\\
= \;& \mathbb{E}_{\tilde{x},a}\left[q_m^{(i)} - q_n^{(i)} | m_{\rho n} > 0\right]\\
= \;& \frac{1}{\mathbb{P}_{\tilde{x},a}(m_{\rho n} > 0)}\mathbb{E}_{\tilde{x},a}\left[(q_m^{(i)} - q_n^{(i)})\mathbf{m}_{(\rho-1)n}(X_n, S_n); m_n > 0\right]\\
\preceq \;& \sqrt{\frac{\rho}{\rho-1}}\frac{1}{h(\tilde{x},a)}\mathbb{E}_{\tilde{x},a}\left[\mathbb{E}\left[(q_m^{(i)} - q_n^{(i)})h(X_n, S_n); m_n > 0 | S_0, ..., S_n\right]\right]
\end{aligned}$$

Since $1_{[m_n>0]}$ and $h(X_n, S_n)$ are $\sigma(S_0, ..., S_n)$-measurable, by (13) and (19), we can observe that

$$
\begin{aligned}
&\mathbb{P}^{(i)}_{\tilde{x},a}(Z_m \neq 0, Z_n = 0 | m_{\rho n} > 0) \\
&\preceq \sqrt{\frac{\rho}{\rho-1}} \frac{1}{h(\tilde{x},a)} \mathbb{E}_{\tilde{x},a}\left[\mathbb{E}\left[(q^{(i)}_m - q^{(i)}_n)|S_0, \dots, S_n\right] h(X_n, S_n); m_n > 0\right] \\
&= \sqrt{\frac{\rho}{\rho-1}} \widehat{\mathbb{E}}_{\tilde{x},a}\left[\mathbb{E}\left[q^{(i)}_m - q^{(i)}_n | S_0, \dots, S_n\right]\right] \\
&= \sqrt{\frac{\rho}{\rho-1}} \widehat{\mathbb{E}}_{\tilde{x},a}\left[q^{(i)}_m - q^{(i)}_n\right] \\
&= \sqrt{\frac{\rho}{\rho-1}} \left[\widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_m \neq 0) - \widehat{\mathbb{P}}^{(i)}_{\tilde{x},a}(Z_n \neq 0)\right].
\end{aligned}
$$

Now let first n and then m tend to $+\infty$. By applying Lemma 3.2, the assertion arrives.

5 Appendix: sketch of the proof of Theorem 2.3

We adapt here the proof of [6] in our setting. Identity (6) may be rewritten as

$$
S_n(\tilde{x}, a) = a + \ln|\tilde{x}R_n| = a + \sum_{k=0}^{n-1} \rho(Y_k) \tag{41}
$$

with $Y_k = (\tilde{x} \cdot R_k, M_k), k \geq 0$. The process $(Y_n)_{n\geq 0}$ is an homogenous Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in the product space $\mathcal{X} = \mathbb{X} \times S_+$, with initial distribution $\delta_{\tilde{x}} \otimes \mu$ and transition operator Q defined by: for any $(\tilde{x}, g) \in \mathcal{X}$ and any bounded Borel function $\varphi : \mathcal{X} \to \mathbb{C}$,

$$
Q\varphi(\tilde{x}, g) := \int_{S^+} \varphi(\tilde{x} \cdot g, h)\mu(dh).
$$

The probability measure $\lambda(d\tilde{x}\ dg) = \nu(d\tilde{x})\ \mu(dg)$ on $\mathcal{X}$ is stationary for the Markov chain $(Y_n)_{n\geq 0}$.

For any $a \in \mathbb{R}$, the sequence $(Y_n, S_n)_{n\geq 0}$ is a Markov chain on $\mathcal{X} \times \mathbb{R}$ whose transition probability $\widetilde{Q}$ is defined by: for any $((\tilde{x}, g), a) \in \mathcal{X} \times \mathbb{R}$ and any bounded Borel function $\psi : \mathcal{X} \times \mathbb{R} \to \mathbb{C}$

$$
\widetilde{Q}\psi((\tilde{x}, g), a) = \int_{S^+} \psi((\tilde{x} \cdot g, h), a + \rho(\tilde{x}, g))\mu(dh).
$$

The operator $\widetilde{Q}_+$ is the restriction of $\widetilde{Q}$ to $\mathcal{X} \times \mathbb{R}^+_*$ and by $\tau := \min\{n \geq 1 : S_n \leq 0\}$. the first time the random process $(S_n)_n$ becomes non-positive.

Denote by $\mathbb{P}_{(\tilde{x},g)}$ the probability measure generated by the finite dimensional distributions of $(Y_n)_{n\geq 0}$ starting at $Y_0 = (\tilde{x}, g) \in \mathcal{X}$ and by $\mathbb{E}_{(\tilde{x},g)}$ the corresponding expectation. Similarly $\mathbb{P}_{(\tilde{x},g),a}$ denotes the probability measure generated by the finite dimensional distributions of $((Y_n, S_n))_{n\geq 0}$ starting at $(Y_0, S_0) = ((\tilde{x}, g), a) \in \mathcal{X}$ and by $\mathbb{E}_{(\tilde{x},g),a}$ the corresponding expectation.

Equality (41) states that $S_n(\tilde{x}, a)$ may be decomposed as a sum of the values of ρ along the trajectories of the Markov chain $(Y_n)_{n\geq 0}$. This is in this context that is stated in [7] a weak invariance principle for a one dimensional Markov walk with a control of the rate of convergence, which is the key ingredient to control the tail of the law of the entrance time in $\mathbb{R}^-$ of the process $(S_n(\tilde{x}, a))_{n\geq 0}$. We emphasize that the quantity $\rho(\tilde{x}, g)$ cannot be expressed in term of the point $\tilde{x} \cdot g$, so that $S_n(\tilde{x}, a)$ may not be decomposed as a sum along the trajectories of $(X_n)_{n\geq 0}$, this explains why we have to introduce the new process $(Y_n)_{n\geq 0}$.

We follow now step by step the approach developed by these authors in the context of product of invertible matrices.

The operator Q acts on the space $\mathcal{C}_b(\mathcal{X})$ of continuous bounded functions $f : \mathcal{X} \to \mathbb{C}$ endowed with the supremum norm

$$|f|_\infty = \sup_{(g,\tilde{x})\in\mathcal{X}} |f(g,\tilde{x})|.$$

In this Appendix, we verify that the restriction of Q (and also a family of pertubations of Q) to some Banach subspace $\mathcal{B} \subset \mathcal{C}_b(\mathcal{X})$ satisfies some spectral gap properties **M1-M3** to be introduced below; for more details we refer to [11]. Under these properties and some additional moment conditions **M4-M5** stated below, we have established in [7] a Komlos-Major-Tusnady type strong approximation result for Markov chains (see Proposition 3.3) which is one of the crucial points in our proof and the one of the main results in [6]. The conditions **M1-M5** also imply the existence of the solution θ of the Poisson equation $\rho = \theta - Q\theta$ which is used in the next section to construct a martingale approximation of the Markov walk $(S_n)_{n\geq 0}$.

Let us now define the Banach space $\mathcal{B}$. For any fix $\epsilon > 0$ and $f \in \mathcal{C}_b(\mathcal{X})$ set

$$k_\varepsilon(f) = \sup_{\substack{\tilde{x}\neq\tilde{y}\\ g\in S_+}} \frac{|f(g,\tilde{x}) - f(g,\tilde{y})|}{d(\tilde{x},\tilde{y})^\epsilon |g|^{4\epsilon}} + \sup_{\substack{\tilde{x}\in\mathbb{X}\\ g\neq h}} \frac{|f(g,\tilde{x}) - f(h,\tilde{x})|}{|g--H|^\epsilon |g|^{3\epsilon}|h|^{3\epsilon}}$$

The space $\mathcal{B} = \mathcal{B}_\varepsilon := \{f \in \mathcal{C}_b : k_\varepsilon(f) < +\infty\}$ endowed with the norm

$$|f|_{\mathcal{B}} = |f|_\infty + k_\varepsilon(f) \tag{42}$$

the space $\mathcal{B}$ becomes a Banach space and also a Banach algebra. Denote by $\mathcal{B}' = \mathcal{L}(\mathcal{B},\mathbb{C})$ the topological dual of $\mathcal{B}$ equipped with the dual norm $|\cdot|_{\mathcal{B}'}$.

Using the techniques of the paper [11], it can be checked that under H1–H3 the condition **M1** below is satisfied:

M1 (Banach space):

i) *The constant functions belongs to* $\mathcal{B}$.

ii) *For every* $(\tilde{x}, g) \in \mathcal{X}$, *the Dirac measure* $\delta_{(\tilde{x},g)}$ *belongs to* $\mathcal{B}'$ *and its norm is* ≤ 1.

iii) $\mathcal{B} \subseteq L^1(\mathbf{Q}((\tilde{x}, g), \cdot)$ *for every* $(\tilde{x}, g) \in \mathcal{X}$

iv) *There exists a constant* $\eta_0 \in (0, 1)$ *such that for any* $t \in [-\eta_0, \eta_0]$ *and* $f \in \mathcal{B}$ *the function* $e^{it\rho} f$ *belongs to* $\mathcal{B}$.

Condition **M1** iii) implies that $\mathbf{Q}f$ is well defined for any $f \in \mathcal{B}$; it follows from **M1** iv) that the perturbed operator $\mathbf{Q}_t f = \mathbf{Q}(e^{it\rho} f)$ is also well defined on $\mathcal{B}$ for any $t \in [-\eta_0, \eta_0]$.

Combining techniques from [11] with the contraction property 2 in Proposition 2.1, on can check that the following conditions **M2-M3** are satisfied:

M2 (Spectral gap): *The operator* Q *on* $\mathcal{B}$ *may be decomposed as* $Q = \Pi + R$ *where* Π *is a one dimensional projector on the constant functions space and* R *is an operator on* $\mathcal{B}$ *with spectral radius* < 1

Notice that $\Pi f = \lambda(f)\mathbf{1}$ for any $f \in \mathcal{B}$.

M3 (Perturbate transition operator): *There exists a constant* $C = C_Q > 0$ *such that*

$$\forall n \geq 1, \forall t \in [-\eta_0, \eta_0] \qquad |Q_t^n|_{\mathcal{B}} \leq C.$$

Using H1, we readily deduce the conditions **M4-M5** below:

M4 (Moment condition): *For any* $p > 2$

$$\sup_{(\tilde{x},g)\in\mathcal{X}} \sup_{n\geq 1} \mathbb{E}^{1/p}_{(\tilde{x},g)} |\rho(Y_n)|^p < +\infty.$$

M5: *The stationary probability measure* λ *satisfies* $\int \sup_{n\geq 0} Q^n \rho^2(\tilde{x}, g)\lambda(dg\ d\tilde{x}) < +\infty.$

In summary, under hypotheses H1, H2, H3 and H4, the conditions **M1-M5** are satisfied. The proof of Theorem 2.3 is decomposed in several steps.

First **M1-M3** allows to construct a martingale approximation of the sequence $(M_n(\tilde{x}, a))_{n\geq 0}$ base on the existence of a solution in $\mathcal{B}$ of the Poisson equation $\rho = (I - Q)\varphi$ (see [6] section 4).

Conditions **M1-M4** yields to the fact that the function V on $\mathcal{X} \times \mathbb{R}_+^*$ defined by

$$V((\tilde{x}, g), a) = \lim_{n\to+\infty} \mathbb{E}_{((\tilde{x},g),a)} (S_n; \tau > n)$$

for any $(\tilde{x}, g) \in \mathcal{X}$ and $a > 0$, is $\widetilde{Q}_+$–Harmonic. Comparing the forms of the operators P and Q implies that $V((\tilde{x}, g), a) = h(\tilde{x} \cdot g, a + \ln|\tilde{x}g|)$. Hypothesis H5 is needed to prove that this function V does not vanish on $\mathcal{X} \times \mathbb{R}^+$.

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